hep-th/9510131
SINP-TNP/95-15

Exact Bethe ansatz solution of nonultralocal quantum mKdV model

Anjan Kundu

Theory Group, Saha Institute of Nuclear Physics,
1/AF Bidhan Nagar, Calcutta 700 064, India.
e-mail: anjan@saha.ernet.in

Abstract

A lattice regularized Lax operator for the nonultralocal modified Korteweg de Vries (mKdV) equation is proposed at the quantum level with the basic operators satisfying a q-deformed braided algebra. Finding further the associated quantum R and Z-matrices the exact integrability of the model is proved through the braided quantum Yang–Baxter equation, a suitably generalized equation for the nonultralocal models. Using the algebraic Bethe ansatz the eigenvalue problem of the quantum mKdV model is exactly solved and its connection with the spin-$\frac{1}{2}$ XXZ chain is established, facilitating the investigation of the corresponding conformal properties.

1 Introduction

The theory of quantum integrable systems made a major breakthrough with the formulation of quantum inverse scattering method (QISM) and the algebraic Bethe ansatz [1]. These techniques enabled us to solve exactly the eigenvalue problem of a number of quantum models including the spin chains [2] as well as the field theoretic models [3]- [6]. However, this success apart from some recent developments was confined mostly to a limited class of models, known as *ultralocal* models. For such systems the Lax operators at different lattice points $j, k; j \neq k$ must commute, i.e.

$$L_{bk}(\mu)L_{aj}(\lambda) = L_{aj}(\lambda)L_{bk}(\mu), \quad a \neq b, \tag{1.1}$$

where a, b signify associative space indices. Due to this constraint the basic equation given by the quantum Yang-Baxter equation (QYBE)

$$R_{ab}(\lambda - \mu)L_{aj}(\lambda)L_{bj}(\mu) = L_{bj}(\mu)L_{aj}(\lambda)R_{ab}(\lambda - \mu), \quad j = 1, \ldots, N \tag{1.2}$$

can be multiplied for different j's and raised to its global form

$$R_{ab}(\lambda - \mu)T_a(\lambda)T_b(\mu) = T_b(\mu)T_a(\lambda)R_{ab}(\lambda - \mu) \tag{1.3}$$

through the monodromy matrix

$$T_a(\lambda) = L_{aN}(\lambda)L_{a,N-1}(\lambda)\ldots L_{a1}(\lambda). \tag{1.4}$$

Taking trace of (1.3) one derives in turn the relation $[trT(\lambda), trT(\mu)] = 0$, and since $\ln(trT(\lambda)) = \sum_{n=1}^{\infty} \frac{C_n}{\lambda^n}$ generates the conserved quantities, one gets $[C_n, C_m] = 0$, for $n \neq m$. That is the infinite number of conserved quantities $\{C_n\}$ are in involuion, which ensures the integrability of the corresponding quantum system represented by the Lax operator L_i.

However, since the above formulation of quantum integrability depends crucially on the ultralocality condition (1.1), it fails for the *nonultralocal* models like WZWN, nonlinear σ-model, Korteweg de Vies (KdV), modified KdV (mKdV) models etc. not satisfying (1.1). The Lax operators L_i of such models involve discrete fields not commuting at different lattice points, which at the continuuum limit may correspond to commutation relaions like

$$[v(x), v(y)] = \delta'(x - y)$$

expressed through derivatives of the δ-function [7].

Nevertheless there were some progress in treating a few of such nonultralocal models [8]- [13] as well as in formulating the basic theory [14]- [17], though the situation in dealing with genuine quantum integrable models is still far from its total perfection. In this context solving quantum KdV and mKdV models at par with the elegant approach of QISM [1, 3] remained as a major challenge.

Our aim here precisely is to address this problem by presenting a quantum mKdV model and solving its eigenvalue equation through algebraic Bethe ansatz, much in parallel to the well established treatment of sine-Gordon [3] or Liuoville [4] model. It should be mentioned that in a very recent work an indigenous nonultralocal formulation of KdV equation as a quantum mapping was given by Volkov [18].

2 Classical mKdV model

Before presenting the quantum mKdV model let us look briefly into some of its classical aspects as a well known integrable system [19] given by the field equation

$$\pm v_t(t,x) + v_{xxx}(t,x) - 6v^2(t,x)v_x(t,x) = 0, \tag{2.1}$$

and possessing an infinite set of conserved quantities $C_n,\ n = 1,2,\ldots$ including the Hamiltonian

$$H = C_3 = \frac{1}{2}\int_{-\infty}^{\infty} dx\left(v_x^2 + v^4\right). \tag{2.2}$$

The Poisson bracket (PB) structure for the field is given in a noncanonical form

$$\{v(x), v(y)\} = \mp\delta'(x-y), \tag{2.3}$$

using which equation (2.1) can be derived from (2.2) as a Hamilton equation. Different signs in equation (2.1) corresponds to fields with different signs in their PB relations (2.3). Note that this noncanonical Poisson bracket (2.3) is the source of the nonultralocal behavior of the mKdV model at the quantum level.

As in all classical integrable systems the nonlinear equation (2.1) can also be obtained from the linear system $\Phi_x = U\Phi, \quad \Phi_t = V\Phi$ as a flatness condition

$$U_t - V_x + [U,V] = 0$$

with the Lax pair

$$U(v,\xi) = i\begin{pmatrix} -\xi & v \\ -v & \xi \end{pmatrix}, \tag{2.4}$$

and

$$V(v,\xi) = \mp i\begin{pmatrix} 4i\xi^3 + 2\xi v^2 & -4i\xi^2 v + 2i\xi v_x + iv_{xx} - 2iv^3 \\ 4i\xi^2 v + 2i\xi v_x - iv_{xx} + 2iv^3 & -4i\xi^3 - 2\xi v^2 \end{pmatrix}. \tag{2.5}$$

It should be remembered that among this pair the Lax operator $U(v,\xi)$ plays more important role and can generate alone all conserved quantities, which with the subsequent application of the PB derives the given equation from the Hamiltonian and other hierachies from the higher conserved quantities. The PB structure associated with the Lax operator also carries important information about the classical r-matrix.

There exists an important connection between mKdV (2.1) and the KdV equation

$$\pm u_t(t,x) + u_{xxx}(t,x) - 6u(t,x)u_x(t,x) = 0, \tag{2.6}$$

through Miura transformation $\quad u = v_x + v^2 \quad$ which maps the PB relation (2.3) for $\quad u(x) = 2\hbar\sum_n l_n e^{-inx} \quad$ to the classical Virasoro algebra

$$i\{l_n, l_m\} = (m-n)l_{n+m} + \frac{n^3}{4\hbar}\delta_{n+m,0} \tag{2.7}$$

3 Lattice regularized quantum Lax operator

We will be interested in the quantum generalization of the mKdV model. However our point of focus will be not the operator evolution equation, but the algebraic aspect of the problem related to its eaxct quantum integrability. For this purpose we consider a lattice regularized Lax operator

$$L_k(\xi) = \begin{pmatrix} (W_k^-)^{-1} & i\Delta\xi W_k^+ \\ -i\Delta\xi(W_k^+)^{-1} & W_k^- \end{pmatrix}, \tag{3.1}$$

involving basic quantum operators $W_k^\pm$, which satisfy an interesting q-deformed algebra

$$\begin{aligned} W_{j+1}^\pm W_j^\pm &= q^{\pm\frac{1}{2}} W_j^\pm W_{j+1}^\pm, \qquad & W_{j+1}^\pm W_j^\mp &= q^{\mp\frac{1}{2}} W_j^\mp W_{j+1}^\pm \\ W_j^+ W_j^- &= q W_j^- W_j^+, \qquad & [W_j^\pm W_j^\pm] &= 0, \end{aligned} \tag{3.2}$$

with $q = e^{i\hbar\gamma}$, similar to the deformed braided algebra [20]. For checking that at the continuum limit one recovers the relations relevant to the mKdV fields, we introduce operators $W_j^\pm = e^{iv_j^\pm}$ with the commutators

$$[v_k^\pm, v_l^\pm] = \mp i\gamma\frac{\hbar}{2}(\delta_{k-1,l} - \delta_{k,l-1}), \tag{3.3}$$

and

$$[v_k^+, v_l^-] = i\gamma\frac{\hbar}{2}(\delta_{k-1,l} - 2\delta_{k,l} + \delta_{k,l-1}), \tag{3.4}$$

which are consistent with algebra (3.2).

At lattice constant $\Delta \to 0$, the discrete operators $v_k^\pm$ go to the quantum field $\Delta v^\pm(x)$ and due to the limit $\partial_x \frac{\delta_{kl}}{\Delta} \to \delta_x(x-y)$ the relation (3.3) reduces clearly to the nonultralocal commutation relation

$$[v^\pm(x), v^\pm(y)] = \pm i\gamma\frac{\hbar}{2}(\delta_x(x-y) - \delta_y(x-y)) = \pm i\gamma\hbar\delta'(x-y) \tag{3.5}$$

recovering at the classical limit the PB relation (2.3) with $\pm$ signs corresponding to the fields $v^\pm(x)$. The reduction of operators $W_j^\pm \approx \mathbf{I} + i\Delta v^\pm(x)$ at the continuum limit yields from (3.1)

$$L_k(v^-, v^+, \xi) \to \mathbf{I} + \Delta\mathcal{L}(v^-, \xi) + O(\Delta^2(v^+))$$

with

$$\mathcal{L}(v^-, \xi) = -i(v^-(x)\sigma^3 + \xi\sigma^2) \tag{3.6}$$

linked to the well known mkdv Lax operator (2.4) as $U(v,\xi) = \sigma^1\mathcal{L}(v^-,\xi)$ involving field $v(x)$, which is related to equation (2.1) with $+$ and PB (3.3) with $-$ signs. It is remarkable that though the lattice Lax operator (3.1) contains both $v^\pm$ variables having nontrivial commutation relation (3.4), the dependence on one of them drops out from the continuum Lax operator (in the present case it is v^+). Moreover since (3.4) at $\Delta \to 0$ yields $[v^+(x), v^-(y)] = \Delta \cdot \partial_{xx}\delta(x-y) \to 0$, the fields answering to different signs of mKdV equation and the PB relations get decouppled at the continuum limit. Note that interchanging the operators $W^+ \leftrightarrow$

W^- along with $q \leftrightarrow q^{-1}$, keeps the underlying algebra (3.2) invariant, while the corresponding Lax operator is changed as $L_k(v^-, v^+, \xi) \to L_k(v^+, v^-, \xi)$, with the continuum limit

$$L_k(v^+, v^-, \xi) \to \mathbf{I} + \Delta \mathcal{L}(v^+, \xi) + O(\Delta^2(v^-))$$

where $\mathcal{L}(v^+, \xi)$ dependents now on the complementary field v^+ related to the $-$ sign of (2.1). We suspect that this feature is likely to be present in the lattice regularization of all such nonultralocal nonrelativistic models exhibiting q-deformed algebra.

4 Braided QYBE and quantum integrability of mKdV model

For establishing the quantum integrability of the model associated with the Lax operator (3.1) we have to generalize the QYBE (1.2) to its braided variant [15]

$$R_{ab}(\lambda - \mu) Z_{ba}^{-1} L_{aj}(\lambda) L_{bj}(\mu) = Z_{ab}^{-1} L_{bj}(\mu) L_{aj}(\lambda) R_{ab}(\lambda - \mu), \quad j = 1, \ldots, N \tag{4.1}$$

complemented at the same time by the braiding relation

$$L_{bj}(\mu) L_{aj+1}(\lambda) = L_{aj+1}(\lambda) Z_{ab}^{-1} L_{bj}(\mu), \quad [L_{bj}(\mu), L_{ai}(\lambda)] = 0, \quad i > j+1, \tag{4.2}$$

which reflects the nonultralocal nature due to the noncommutation of the operators at neighboring lattice points. In such nonultralocal models along with the standard R-matrix solution of the Yang–Baxter equation (YBE)

$$R_{12}(\lambda - \mu) R_{13}(\lambda) R_{23}(\mu) = R_{23}(\mu) R_{13}(\lambda) R_{12}(\lambda - \mu) \tag{4.3}$$

one needs to find a Z-matrix satisfying similar equations [15]

$$R_{21}(\lambda) Z_{13} Z_{23} = Z_{23} Z_{13} R_{21}(\lambda) \quad Z_{12} Z_{13} R_{32}(\lambda) = R_{32}(\lambda) Z_{13} Z_{12} \tag{4.4}$$

and

$$Z_{12} Z_{13} Z_{23} = Z_{23} Z_{13} Z_{12}. \tag{4.5}$$

In general the Z-matrix may also depend on the spectral parameters. Note that the set of YBE's (4.3)-(4.5) is compatible with the associativity condition of (4.1). This formulation is specialized to handle the nonultralocality not extending beyond the next neighbouring points, which agrees with the present mKdV model with the field commutator (3.5).

We may define the monodromy matrix in the interval as usual by

$$T_a^{[k,j]}(\lambda) = L_{ak}(\lambda) L_{a,k-1}(\lambda) \ldots L_{aj}(\lambda)$$

acting in the space $\mathcal{H} \equiv V_k \otimes V_{k-1} \otimes \ldots \otimes V_j$. It is important to note that due to the braided QYBE (4.1) and braiding relations (4.2) the transition to the global equation (in analogy with the transition from (1.2) to (1.3)) is possible also for this nonultralocal case resulting

$$R_{12}(\lambda-\mu)Z_{21}^{-1}T_1^{[k,j]}(\lambda)T_2^{[k,j]}(\mu)=Z_{12}^{-1}T_2^{[k,j]}(\mu)T_1^{[k,j]}(\lambda)R_{12}(\lambda-\mu) \tag{4.6}$$

where the similarity in form with (4.1) reflects the underlying Hopf algebra structure.

For physical models with periodic boundary condition $L_{aj+N}(\lambda)=L_{aj}(\lambda)$ more interesting thing happens to the global monodromy matrix $T_a(\lambda)\equiv T_a^{[N,1]}(\lambda)$ for the closed chain $[N,1]$, since now $L_1(\lambda)$ and $L_N(\lambda)$ becomes nearest neighbours again with nontrivial commutation relation (4.2). The corresponding global braided QYBE is given consequently by

$$R_{12}(\lambda-\mu)Z_{21}^{-1}T_1(\lambda)Z_{12}^{-1}T_2(\mu)=Z_{12}^{-1}T_2(\mu)Z_{21}^{-1}T_1(\lambda)R_{12}(\lambda-\mu) \tag{4.7}$$

Equipped with this theory for quantum nonultralocal models we may start now with the lattice Lax operator (3.1) of the mKdV model, which from the braiding relation (4.2) and with the use of commutators (3.2) for neighboring points yields a solution

$$Z_{12}=Z_{21}=q^{-\sigma^3\otimes\sigma^3}. \tag{4.8}$$

For finding the quantum R-matrix we plug Z-matrix (4.8) together with L_k in the braided QYBE (4.1) and use relation like

$$(W_j^+)^\alpha(W_j^-)^\beta=q^{\alpha\beta}(W_j^-)^\beta(W_j^+)^\alpha$$

for the same lattice point along with the trivial one

$$[(W_j^\pm)^\alpha(W_j^\pm)^\beta]=0.$$

After some algebra we arrive at the remarkable result that the R-matrix of the lattice mKdV model is given by the same trigonometric solution

$$R(\lambda-\mu)=\begin{pmatrix} a(\lambda-\mu) & & & \\ & b(\lambda-\mu) & c(\lambda-\mu) & \\ & c(\lambda-\mu) & b(\lambda-\mu) & \\ & & & a(\lambda-\mu)\end{pmatrix} \tag{4.9}$$

with

$$a(\lambda-\mu)=\sin(\lambda-\mu+\hbar\gamma),\quad b(\lambda-\mu)=\sin(\lambda-\mu),\quad c(\lambda-\mu)=\sin(\hbar\gamma) \tag{4.10}$$

as that of the XXZ-spin chain. It is an easy check that (4.9) and (4.8) thus found satisfy all YBE's (4.3-4.5) and therefore, may be used for the braided QYBE's (4.1) and (4.7) applicable to the periodic quantum mKdV model. Moreover due to the fact that R-matrix (4.9) commutes with Z-matrix (4.8), the braided QYBE's in the present case simplify further to

$$R_{12}(\lambda-\mu)L_{1j}(\lambda)L_{2j}(\mu)=L_{2j}(\mu)L_{1j}(\lambda)R_{12}(\lambda-\mu) \tag{4.11}$$

and

$$R_{12}(\lambda-\mu)T_1(\lambda)Z_{12}^{-1}T_2(\mu)=T_2(\mu)Z_{12}^{-1}T_1(\lambda)R_{12}(\lambda-\mu), \tag{4.12}$$

respectively.

Before proceeding further we must justify our choice of the monodromy matrix taken in the same form as in the ultralocal case:

$$T_a(\lambda) = L_{aN}(\lambda)\dots L_{a,k+1}(\lambda)L_{a,k}(\lambda)\dots L_{a1}(\lambda). \tag{4.13}$$

Evidently this is possible when the nonultralocal property does not spoil the detailed structure of this global object and one gets $L_{k+1}L_k(\xi) \sim L_kL_{k+1}(\xi)$. From (3.1) we obtain the explicit form

$$L_{k+1}L_k(\xi) = \left(\begin{array}{cc} L_{11}^{(k+1,k)} & L_{12}^{(k+1,k)} \\ L_{21}^{(k+1,k)} & L_{22}^{(k+1,k)} \end{array} \right) \tag{4.14}$$

where

$$\begin{array}{rcl} L_{11}^{(k+1,k)} & = & (W_{k+1}^-)^{-1}(W_k^-)^{-1} + (\Delta\xi)^2 W_{k+1}^+(W_k^+)^{-1} \\ L_{22}^{(k+1,k)} & = & W_{k+1}^- W_k^- + (\Delta\xi)^2 (W_{k+1}^+)^{-1} W_k^+ \\ L_{12}^{(k+1,k)} & = & i\Delta\xi((W_{k+1}^-)^{-1}W_k^+ + W_{k+1}^+ W_k^- \\ L_{21}^{(k+1,k)} & = & -i\Delta\xi((W_{k+1}^+)^{-1}(W_k^-)^{-1} + W_{k+1}^-(W_k^+)^{-1}) \end{array} \tag{4.15}$$

Using nonultralocal relations (3.2) one shows that for all expressions (4.15), $L_{ij}^{(k+1,k)} = q\ L_{ij}^{(k,k+1)}$ hold, which leads to the conclusion $L_{k+1}L_k(\xi) = q\ L_kL_{k+1}(\xi)$ justifying the choice (4.13).

For identifying the commuting set of conserved quantities from (4.7) one may find the k-trace following the prescription of [21, 15]. However since in our case the R-matrix commutes with Z we may start from (4.12), multiply it by Z and take the trace as

$$tr(Z_{12}T_1(\lambda)Z_{12}^{-1}T_2(\mu)) = tr(Z_{12}T_2(\mu)Z_{12}^{-1}T_1(\lambda)),$$

forcing the standard relation

$$[trT(\lambda), trT(\mu)] = 0 \quad .$$

Therefore for

$$T(\lambda) = \left(\begin{array}{cc} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{array} \right), \tag{4.16}$$

$\tau(\lambda) = trT(\lambda) = A(\lambda) + D(\lambda)$ becomes the generator of the conserved quantities with the Hamiltonian of the mkdV system produced as $\partial_\lambda^{'''} ln\tau(\lambda) \mid (\lambda = \infty) = C_3 = H$ at the continuum limit. Note that since the symmetry of the R-matrix allows $[R_{12}, k_1k_2] = 0$ with $k = q^{-\frac{\kappa}{2}\sigma^3}$, one may also consider

$$\tau(\lambda) = tr(kT(\lambda)) = q^{-\frac{\kappa}{2}}A(\lambda) + q^{\frac{\kappa}{2}}D(\lambda)$$

also for generating the commuting family of conserved quantities.

5 Algebraic Bethe ansatz solution for the eigenvalue problem

For solving the eigenvalue problem of the quantum mKdV model

$$\begin{aligned}\tau(\lambda) \mid \lambda_1, \lambda_2, \cdots, \lambda_m > &= (q^{-\frac{\kappa}{2}} A(\lambda) + q^{\frac{\kappa}{2}} D(\lambda)) \mid \lambda_1, \lambda_2, \cdots, \lambda_m > \\ &= \Lambda(\lambda) \mid \lambda_1, \lambda_2, \cdots, \lambda_m > \end{aligned} \tag{5.1}$$

including that of its energy spectrum we may proceed in a way similar to the standard algebraic Bethe ansatz formalism [1], keeping at the same time the track of the nonultralocal signatures. Considering $B(\lambda)$ and $C(\lambda)$ as the *creation* and *anihilation* operators of the pseudoparticles we assume the eigenvectors in the usual Bethe ansatz form

$$\mid \lambda_1, \lambda_2, \cdots, \lambda_m >= B(\lambda_1)B(\lambda_2)\ldots B(\lambda_m) \mid 0 > \tag{5.2}$$

where $\mid 0 >$ is the pseudovacuum. For calculating the eigenvalue of $\tau(\lambda) = q^{-\frac{\kappa}{2}} A(\lambda) + q^{\frac{\kappa}{2}} D(\lambda)$ we need to know first the commutation relations of $A(\lambda)$ and $D(\lambda)$ with $B(\mu)$, which may be obtained from the braided QYBE (4.12) using the R, Z-matrices given as (4.9) and (4.8):

$$\begin{aligned}A(\lambda)B(\mu) &= q^{-2}\frac{a(\mu-\lambda)}{b(\mu-\lambda)}B(\mu)A(\lambda) - q^{-2}\frac{c(\mu-\lambda)}{b(\mu-\lambda)}A(\mu)B(\lambda) \\ D(\lambda)B(\mu) &= q^{2}\frac{a(\lambda-\mu)}{b(\lambda-\mu)}B(\mu)D(\lambda) - q^{2}\frac{c(\lambda-\mu)}{b(\lambda-\mu)}D(\mu)B(\lambda)\end{aligned} \tag{5.3}$$

Note that the multiplicative factors $q^{\pm 2}$ appearing in the above expressions is the contribution from nonultralocality. For proceeding further along the QISM line one must define a pseudovacuum such that

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \mid 0 >= \begin{pmatrix} \alpha^N(\lambda) & * \\ 0 & \beta^N(\lambda) \end{pmatrix} \mid 0 >, \tag{5.4}$$

which corresponds to the action of L_i on local $\mid 0 >_i$, where $\mid 0 >= \prod_{i=1}^N \mid 0 >_i$ as $L_i \mid 0 >_i = \begin{pmatrix} \alpha(\lambda) & * \\ 0 & \beta(\lambda) \end{pmatrix} \mid 0 >_i$. That is the pseudovacuum must be the eigenstate of the diagonal elements of L_i and anihilated by one of its off-diagonal elements.

However as evident from the form of L_k (3.1) naive application of a single Lax operator can not construct such a pseudovacuum. The situation is more like the Toda chain [22] or the lattice Liouville model [4]. Therefore one may either follow the functional Bethe ansatz method formulated in [22] or the remedy suggested in [4]. We adopt here the brilliant solution of [4] by considering the action of a *fused* Lax operator $L^{(2)} = L_k L_{k+1}$ in constructing the pseudovacuum.

A careful analysis involving the action of the fused Lax operator elements (4.15) shows that here the pseudovacuum may be chosen in the form

$$\mid 0 >= \prod_{(\frac{k+1}{2}=1)}^{N} \mid 0 >_k$$

where

$$| \, 0 >_k = \delta(W^+_{k+1} - qW^+_k)\delta(W^-_{k+1} + (W^-_k)^{-1}). \tag{5.5}$$

For checking this claim and deriving the required relation (5.4) we consider the action of $L^{(k+1,k)}_{ij}$ from (4.15) on (5.5). This gives for the diagonal elements

$$\begin{aligned} L^{(k+1,k)}_{11} \, | \, 0 >_k &= \Big((W^-_{k+1})^{-1}(W^-_k)^{-1} + (\Delta\xi)^2 W^+_{k+1}(W^+_k)^{-1}\Big) \, | \, 0 >_k \\ &= -\Big(1 - (\Delta\xi)^2 q\Big)\, \delta(W^+_{k+1} - qW^+_k)\delta(W^-_{k+1} + (W^-_k)^{-1}) \end{aligned} \tag{5.6}$$

i.e.

$$\alpha(\lambda) = 2ie^{i(\lambda + \frac{\gamma\hbar}{2})} \sin(\lambda + \frac{\gamma\hbar}{2}) \tag{5.7}$$

with $\Delta\xi = e^{i\lambda}$ and similarly

$$\begin{aligned} L^{(k+1,k)}_{22} \, | \, 0 >_k &= \Big(W^-_{k+1}W^-_k + (\Delta\xi)^2 (W^+_{k+1})^{-1} W^+_k\Big) \, | \, 0 > \\ &= -\Big(1 - (\Delta\xi)^2 q^{-1}\Big) \, | \, 0 >_k \end{aligned} \tag{5.8}$$

yielding

$$\beta(\lambda) = 2ie^{i(\lambda - \frac{\gamma\hbar}{2})} \sin(\lambda - \frac{\gamma\hbar}{2}). \tag{5.9}$$

On the other hand for the off-diagonal element we obtain

$$\begin{aligned} L^{(k+1,k)}_{21} \, | \, 0 > &= -i\Delta\xi \Big((W^+_{k+1})^{-1}(W^-_k)^{-1} + W^-_{k+1}(W^+_k)^{-1}\Big) \, | \, 0 >_k \\ &= -i\Delta\xi \Big(q^{-1}(W^+_k)^{-1}(W^-_k)^{-1} - (W^-_k)^{-1}(W^+_k)^{-1}\Big) \, | \, 0 >_k = 0 \end{aligned} \tag{5.10}$$

using the relation $(W^-_k)^{-1}(W^+_k)^{-1} = q^{-1}(W^+_k)^{-1}(W^-_k)^{-1}$, which corresponds to the anihilation operation. The other element however gives $L^{(k+1,k)}_{12} \, | \, 0 >_k \neq 0$. The eigenvalue problem (5.1) of the mKdV model therefore may be solved using the relations (5.3) taking into account the pseudovacuum properties (5.4). Finally inserting the results (5.7) and (5.9) one obtains

$$\begin{aligned} \Lambda(\lambda) &= q^{-\frac{\kappa}{2}}\alpha(\lambda)^N \prod_j \left(q^{-2}\frac{a(\lambda_j - \lambda)}{b(\lambda_j - \lambda)}\right) + q^{\frac{\kappa}{2}}\beta(\lambda)^N \prod_j \left(q^2 \frac{a(\lambda - \lambda_j)}{b(\lambda - \lambda_j)}\right) \\ &= 2ie^{i\lambda}[q^{-\frac{\kappa}{2}+\frac{N}{2}-2m} \sin^N(\lambda + \frac{\gamma\hbar}{2}) \prod_{j=1}^m \frac{\sin(\lambda - \lambda_j - \gamma\hbar)}{\sin(\lambda - \lambda_j)} \\ &+ q^{-(-\frac{\kappa}{2}+\frac{N}{2}-2m)} \sin^N(\lambda - \frac{\gamma\hbar}{2}) \prod_{j=1}^m \frac{\sin(\lambda - \lambda_j + \gamma\hbar)}{\sin(\lambda - \lambda_j)}] \end{aligned} \tag{5.11}$$

The determining equation for the rapidity parameters λ_j may be obtained from the vanishing residue condition of the eigenvalue equation (5.11) in the usual way:

$$q^{-\kappa+N-4m}\left(\frac{\sin(\lambda + \frac{\gamma\hbar}{2})}{\sin(\lambda - \frac{\gamma\hbar}{2})}\right)^N = \prod_{j\neq k}^m \left(\frac{\sin(\lambda - \lambda_j + \gamma\hbar)}{\sin(\lambda - \lambda_j - \gamma\hbar)}\right) \tag{5.12}$$

Note that the nonultralocal property of the model is reflected in the q^{-4m} factors appearing in the above relations. Such factors were found to appear also in ultralocal models with twisted boundary conditions [23, 24]. However for the mKdV model considered here, they depend on the total number of lattice points $2N$ as in [4] and more interestingly, on the number of pseudoparticle excitations m. Comparison of (5.12) with the Bethe equation for the spin-$\frac{1}{2}$ XXZ chain with twisted boundary condition

$$q^{-\kappa}\left(\frac{\sin(\lambda+\frac{\gamma\hbar}{2})}{\sin(\lambda-\frac{\gamma\hbar}{2})}\right)^N=\prod_{j\neq k}^{m}\left(\frac{\sin(\lambda-\lambda_j+\gamma\hbar)}{\sin(\lambda-\lambda_j-\gamma\hbar)}\right) \tag{5.13}$$

reveals the startling fact that, if the extra q factors are ignored the mKdV model on a lattice with $2N$ points becomes equivalent to the spin-$\frac{1}{2}$ XXZ chain with total lattice points N. Of course the hamiltonian of the respective systems corresponds to different conserved quantities and their expansion point of spectral parameters for generating conservation laws also differs. For obtaining the mKdV field model one has to consider ultimately the continuum limit.

6 Concluding remarks

The spin-$\frac{1}{2}$ XXZ chain is known to be connected with the six-vertex model, a classical statistical model in 2-dimrnsions. Therefore since the finite-size corrections and the conformal properties of the six-vertex model is a well studied subject, the intriguing relation between the quantum mKdV and the spin-$\frac{1}{2}$ XXZ chain found here should be an important result in analyzing the conformal properties of the mKdV model with $q=e^{i\frac{\pi}{\nu+1}}$. Let us recall that the six-vertex model (with a seam) yields [24] for the ground state energy

$$E_0=Nf_\infty-\frac{1}{N}\frac{\pi}{6}c+O(\frac{1}{N^2})$$

and for the excited states

$$E_m-E_0=\frac{2\pi}{N}(\Delta+\tilde{\Delta})+O(\frac{1}{N^2}) \qquad P_m-P_0=\frac{2\pi}{N}(\Delta-\tilde{\Delta})+O(\frac{1}{N^2})$$

where $c=1-\frac{6}{\nu(\nu+1)}$, $\nu=2,3,\ldots$ for $\kappa=1$ is the central charge and $\Delta,\tilde{\Delta}$ are conformal weights of unitary minimal models of the corresponding conformal field theory.

The conformal properties of the mKdV model have enhanced importance due to its mapping into the KdV model through Miura transformation. The KdV field in turn is related to the generators of the Virasoro algebra answering to the 2-d conformal transformations [25, 26]. In this context the continuum limit of the fused Lax operator (4.14), which plays a crucial role in constructing the pseudovacuum and deriving the Bethe equation (5.12), is also worth analyzing. The linear system corresponding to this Lax operator is $\Phi_{k+2}(\xi)=L^{(2)}(\xi)\Phi_k(\xi)$, which at the continuum limit would yield the relations

$$\Phi_x(0)=-\frac{i}{2}(v^-(x)\sigma^3)\Phi(0), \quad \text{and} \quad \Phi_{xx}(0)=\begin{pmatrix} u(x) & \\ & \tilde{u}(x)\end{pmatrix}\Phi(0) \tag{6.1}$$

where $\Phi(0) = \Phi(\xi = 0)$ and the fields $u, \tilde{u}$ are given by

$$u(x) = iv_x^-(x) + v^{-2}(x), \quad \tilde{u}(x) = -iv_x^-(x) + v^{-2}(x) \tag{6.2}$$

i.e. they represent KdV fields related trough Miura transformation. Replacement of $v^- \leftrightarrow v^+$ would generate similarly KdV fields through v^+. Due to such relations between mKdV and KdV fields and their connection with the Liouville field [25], the recently discovered equivalence [4] between quantum lattice Liouville model and spin $(-\frac{1}{2})$ XXZ chain acquires renewed importance.

It is also hoped that the main results of the present work, e.g. construction of the exact lattice Lax operator for the quantum mKdV model, the R and Z matrices as solutions of the braided QYBE, the Bethe ansatz solution of the related eigenvalue problem and finally the mapping between quantum mKdV and spin-$\frac{1}{2}$ XXZ chain, would be of significance in the alternate approach of [26] in formulating CFT through quantum KdV related theory.

References

[1] L. D. Faddeev, Sov. Sc. Rev. C1 (1980) 107.

[2] L. A. Takhtajan and L. D. Faddeev, Russian Math. Survey 34 (1979) 11.

P.P. Kulish and E. K. Sklyanin, Phys. Lett. A70 (1979) 461

[3] E. K. Sklyanin, L. A. Takhtajan and L. D. Faddeev, Teor. Mat. Fiz. 40 (1979) 194.

[4] L. D. Faddeev and O. Tirkkonen, *Connection of the Liouville model and the XXZ spin chain*, preprint hep-th/9506023 (1995)

[5] A. Kundu and B. Basumallick, J. Math. Phys. 34 (1993) 1252.

[6] P. Kulish and E. K. Sklyanin, Lect. Notes in Phys. (ed. J. Hietarinta et al, Springer,Berlin, 1982) vol. 151 p. 61.

[7] J.M. Maillet, Phys. Lett. 162 B (1985) 137;

S. A. Tsyplyaev, Teor. Mat. Fiz. 46 (1981) 24

J.M. Maillet, Nucl. Phys. B269 (1986) 54 ; Phys. Lett. 167 B (1986) 401.

M. Semenov-Tian-Shansky Funct. Anal. Appl. 17 (1983) 259

[8] V. E. Korepin, J. Sov. Math. 23 (1983) 2429.

[9] F.W. Nijhoff, H.W. Capel and V.G. Papageorgiou, *Integrable quantum mappings* , Clarkson Univ. preprint INS 168/91 (1991)

F.W. Nijhoff and H.W. Capel, *Integrable quantum mapping and nonultralocal Yang-Baxter structure*, Clarkson Univ. preprint INS 171/91 (1991).

[10] O. Babelon and L. Bonora, Phys. Lett. 253 B (1991) 365

O. Babelon, Comm. Math. Phys. 139 (1991) 619

L. Bonora and V. Bonservizi, Nucl. Phys. B 390 (1993) 205.

[11] A. Alekseev, L.D. Faddeev, M. Semenov-Tian-Shansky and A. Volkov,*The unravalling of the quantum group structure in the WZNW theory*, preprint CERN-TH-5981/91 (1991)

L. D. Faddeev, Comm. Math. Phys. 132 (1990) 131

A. Alekseev, S. Statashvili, Comm. Math. Phys. 133 (1990) 353

B. Blok, Phys. Lett. 233B (1989) 359

[12] N. Yu. Reshetekhin and M. Semenov-Tian-Shansky , Lett. Math. Phys. 19 (1990) 133

[13] M. Chu, P. Goddard, I. Halliday, D. Olive and A. Schwimmer, Phys. Lett. 266 B (1991) 71.

[14] L. Freidel and J.M. Maillet, Phys. Lett. 262 B (1991) 278.

L. Freidel and J.M. Maillet, Phys. Lett. 263 B (1991) 403

[15] L. Hlavaty and Anjan Kundu, *Quantum integrability of nonultralocal models through Baxterization of quantized braided algebra*, preprint BONN-TH-94-15, hep-th/9406215

[16] L. Hlavaty *Algebraic framework for quantization of nonultralocal models*, preprint hep-th/9412142

C. Schwiebert, *Generalized quantum inverse scattering method* preprint RIMS 1003, hep-th/9412237,

[17] M. Semenov Tian-Shansky and A. Sevostyanov *Classical and quantum nonultralocal systems on the lattice* preprint hep-th/ 9509026 (1995)

[18] A. Yu. Volkov *Quantum lattice kdV equation* preprint hep-th/950924

[19] S. Novikov, V. Manakov, L. Pitaevskii and V. Zakharov, *Theory of Solitons* (Plenum, N.Y., 1984)

[20] S. Majid, J. Math. Phys. 32 (1991) 3246.

L. Hlavatý, J. Math. Phys. 35 (1994) 2560.

[21] E. K. Sklyanin, J. Phys. A 21 (1988) 2375.

[22] E. K. Sklyanin, Lect. Notes in Phys. 226 (Springer,Berline, 1985) p. 196

[23] A. Klümper, M. Batchelor and P. Pearce, J. Phys. A 24 (1991) 3111; A. Klümper, T. Wehner and J. Zittarz, J. Phys. A 26 (1993) 2815

[24] A. Karowski, Nucl. Phys. B 300 [FS 22] (1988) 479

V. de Vega and A. Karowski, Nucl. Phys. B 285 [FS 19] (1987) 619

[25] J. L. Gervais in *DST Workshop on particle physics -supersrting theory* (ed. HS Mani et al, W. Sc., 1988) p. 287 ; J. L. Gervais Comm. Math. Phys. 138 (1991) 301

[26] V. V. Bazhanov, S. Lukyanov and A. B. Zamolodchikov, *Integrable structure of CFT, quantum KdV and therodynamic Bethe ansatz ,* hep-th/9412229.

V. A. Fateev and S. Lukyanov, Int. J. Mod. Phys. A7 (1992) 853, 1325